Pergamon

Int. J. Solids Structures Vol. 34, No. 29, pp. 3773–3780, 1997

0020–7683/97 $17.00 + .00

PII: S0020–7683(97)00020–6

FEATURES OF THE STRESS PATH AT THE POST ADAPTATION STAGE AND RELATED SHAKEDOWN CONDITIONS

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(*Received* 13 *May* 1996; *in revised form* 1 *December* 1996)

Abstract—A new necessary condition for adaptation of elastic plastic bodies with internal variables to cyclic loading is deduced from the features of the deformation path at the post adaptation stage: shakedown is possible for taken values of internal variables, if there exists a solution of a min-max problem. In some cases (perfectly plastic bodies, and isotropically strain hardening) the condition is also sufficient. In the last case the condition provides a method to determining the minimum value of hardening parameter which can be acquired under the given loading program. An example of application of the method is given. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

The basis of the modern shakedown theory is well known to be originally developed by Melan (1938) and Koiter (1960) in the framework of elastic perfectly plastic bodies. A lot of efforts have been devoted to extending the shakedown theory to more realistic material behavior during the last decades. In main, these efforts have been concerned with the strain-hardening phenomenon. Melan (1938) was the first to present the static shakedown theorems for a material model with linear unlimited kinematic strain hardening. Similar assumptions were taken by Maier (1973, 1987) and Ponter (1975). König (1982, 1987), and König and Siemaszko (1988) considered the material model with nonlinear kinematic hardening. Maier and Novati (1990a, b) modeled the general hardening of materials with piece-wise linear yield surfaces. A limited linear kinematic hardening was considered by Weichert and Gross-Weege (1988). They used a two surface yield condition for this purpose. The general nonlinear kinematic hardening phenomenon was taken into account by Stein *et al.* (1992) who employed the so called Overlay Constitutive Material Model.

Invention of the Generalized Standard Material Model (GSMM) (Halphen and Nguen, 1975) provided new opportunities for the shakedown theory extensions to material models with internal variables. Mandel (1976) generalized the static shakedown theorem to linear and unbounded strain-hardening. Polizzotto *et al.* (1991) extended the main achievements of the shakedown theory to GSMM.

Corigliano *et al.* (1995) employed the so called Generalized Nonstandard Material Model (GNMM) with nonassociativity, bounded nonlinear hardening, and the assumption of convexity of plastic potential in stresses and in internal variables to develop kinematic approach to the problems of dynamic shakedown.

The sufficient static (Melan) shakedown theorem was generalized to the GNMMs by Pycko and Maier (1995) under the same assumptions.

Shakedown of elastic perfectly plastic bodies is known to occur due to formation of pertinent fields of residual stresses, whereas the shakedown of bodies with more complex mechanical behavior occurs not only due to residual stresses but also due to changes in their microstructure which change the response of the body to loading. These changes being reflected in the evolution of appropriate chosen internal variables, result in changes in the yield condition. For example the adaptation (shakedown) of the strain hardening elastic plastic bodies to cyclic loading occurs due to strain hardening and formation of the pertinent fields of residual stresses.

The present consideration is restricted by the elastic shakedown.

The approach developed in this paper is based on the concept of the Limit Yield Condition (LYC): the yield condition corresponding to the post-adaptation stage of the deformation process (Druyanov and Roman, 1995, 1996).

The post adaptation stage has been attracting the attention of investigators for a long time (Martin, 1975; Polizzotto, 1994; Polizzotto and Borino, 1996; Gokhfeld and Sadakov, 1995, and others).

The problems of existence of the LYC and its estimating are specific for the shakedown theory in the case of the material models with internal variables. Some computing methods capable of direct computation of the steady cycle parameters have been developed to date. Zarka *et al.* (1978) proposed a method to estimate the steady cycle parameters on the basis of elastic solution and the first cycle of elastic-plastic computation. Jiang and Leckie (1992) generalized Zarka's method to the case where a reverse plastic deformation occurs during the post adaptation stage. Resolving of the problem of evaluating of the steady cycle parameters is facilitated if it is combined with methods of discretization, for example, with the Finite Element Method. (Maier and Novati, 1990a; Stumpf, 1993; Weichert and Gross-Weege, 1988). Gokhfeld and Cherniavsky (1980) reduced the problem of evaluating of the steady cycle parameters to a non-linear optimization problem. Some theorems and methods capable of estimating the LYC for strain hardening bodies were advanced recently by the authors (Druyanov and Roman, 1995, 1996). In the present paper the same problem is investigated from another point of view.

The starting point of the present deduction is the notion that at the post adaptation stage of deformation process (if exists) the stress path in the stress space reaches the limit yield surface, but the stresses do not cause plastic deformation. This is possible under the requirements, that the stress path lies partly on the yield surface, or touches it at isolated instants. These requirements allow to derive a new condition for elastic adaptation which can be realized if the loading program is given as a function of time. If only the bounds of the loading are prescribed then the condition is necessary for shakedown.

The above approach results in a problem of mathematical programming whose solution provides us with a field of residual stresses which corresponds, under a fixed value of the hardening parameter vector, to the minimum value of the yield function at the post adaptation stage. The existence of this field is necessary for adaptation for a wide class of material models with internal variables. It allows to answer the question: whether the body under consideration can adapt itself to the given loading program under the taken value of the internal variables vector. At the same time the existence of the above min-max problem is sufficient for adaptation in the events of perfectly plastic bodies, as well as elastic plastic bodies with limited isotropic strain hardening. In the last event the above solution provides us with a lower estimation of the strain hardening parameter.

Zwolinski and Bielawski (1987), Pycko and Mroz (1992), Zwolinski (1995) brought in the theory the notion of alternative shakedown multiplier. For perfectly plastic and kinematically strain hardening bodies with the homogeneous yield function of the first order in stresses the problem of its determination is reduced to a min-max problem which is like formally the min-max problem derived in the present paper. However, no multipliers is considered in the proposed paper. Its findings are different. Besides, unlike those contributions, the present derivation is not based on the Melan theorem, and is not restricted by any assumptions concerning the yield function and internal variables.

Also, Polizzotto (1993) proposed an heuristic variational method applicable to elastic perfectly plastic bodies which is close to the above mentioned ones.

2. CONSTITUTIVE MATERIAL MODEL

Let $\Phi(\boldsymbol{\sigma}, \boldsymbol{\chi}) = 0$ be the yield condition, $\boldsymbol{\sigma}$ be the stress tensor, and $\boldsymbol{\chi}$ be the set of internal variables which is formally considered here as a vector.

The constitutive equations are written as follows:

$$\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p \tag{1}$$

$$\dot{\varepsilon}^p = \lambda \Phi_{,\sigma}, \dot{\varepsilon}^e = \mathbf{L} : \dot{\sigma} \tag{2}$$

$$\lambda > 0 \text{ if } \Phi = 0 \quad \text{and} \quad d'\Phi = \Phi_{,\sigma} : d\sigma > 0 \tag{3}$$

$$\lambda = 0 \text{ if } \Phi < 0, \text{ or if } \Phi = 0 \quad \text{and} \quad d'\Phi \leqslant 0 \tag{4}$$

$$\dot{\chi} = \mathbf{B} : \dot{\varepsilon}^p \tag{5}$$

where $\dot{\varepsilon}^e$ and $\dot{\varepsilon}^p$ are the elastic and plastic parts of the strain rate tensor $\dot{\varepsilon}$, λ is a non negative scalar factor, $\mathbf{L}$ is the tensor of elastic compliances, $\mathbf{B}(\sigma, \chi)$ is the third order tensor governing the evolution of the internal variables, $\Phi_{,\sigma} = \partial\Phi/\partial\sigma$, $\dot{\varepsilon} = d\varepsilon/dt$, $\mathbf{a} : \mathbf{b} = a_{ij}b_{ij}$, $\mathbf{B} : \dot{\varepsilon}^p = B_{ikl}\dot{\varepsilon}^p_{kl}$; $B_{ikl} = B_{ilk}$.

The consideration is restricted by the condition that the material does not go out of the region of its stability.

Consider the equation $\dot{\Phi} = \Phi_{,\sigma} : \dot{\sigma} + \Phi_{,\chi} \cdot \dot{\chi} = 0$. An application of (2) and (5) yields: $\Phi_{,\sigma} : \dot{\sigma} + \lambda\, \Phi_{,\chi}\mathbf{B} : \Phi_{,\sigma} = 0$. This equation allows to determine λ and exclude it from (2)–(4), if $\Phi_{,\chi} \cdot \mathbf{B} : \Phi_{,\sigma} \neq 0$.

During neutral loading $\Phi_{,\sigma} : \dot{\sigma} = 0$ and $\lambda = 0$ if $\Phi_{,\chi} \cdot \mathbf{B} : \Phi_{,\sigma} \neq 0$. Thus, the material does not suffer any plastic deformation in this case.

It is taken that $\Phi < 0$ corresponds to the interior of the yield surface $\Phi = 0$, i.e., to the elastic part of the stress space σ, and that the point $\sigma = 0$ is always in the interior of the yield surface: $\Phi(0, \chi) < 0$.

The yield condition is assumed to be not concave with respect to σ for all the admissible values of χ, but any convexity of the yield function in χ is not assumed.

3. SOME FEATURES OF THE DEFORMATION PATH AT THE POST ADAPTATION STAGE OF DEFORMATION PROCESS

Two ways to plastic failure under cyclic loading are usually considered: ratchetting (unlimited accumulation of plastic deformation of the same sign) and plastic shakedown (non decreasing alternating plastic deformation). In the both events the total plastic dissipation is unbounded: $D^p \to \infty$ as $t \to \infty$, where

$$D^p = \int_0^t dt \int_Q \sigma : \dot{\varepsilon}^p \, dQ, \tag{6}$$

and Q denotes the part of space occupied by the body.

If D^p is bounded from above ($D^p < \infty$), the body is said to shake down (to adapt itself) to the loading program. Obviously, this definition excludes the so called plastic shakedown out of consideration. Note that $D^p \geqslant 0$ because $\dot{D}^p \geqslant 0$.

According to the accepted constitutive material model, any plastic deformation causes changes in the internal variables (eqn (5)) which, in turn, affect the yield function.

If irreversible changes in the material micro structure cease in a limited time (t^0), the shakedown is called elastic. In that event, only elastic deformation occurs from time t^0 on, and the entire deformation process can be divided into adaptation (transient) and post-adaptation (stationary) stages. Otherwise, if D^p is limited, but t^0 is unlimited, the shakedown could be named asymptotic.

If the elastic shakedown occurs, the internal variables tend to definite limit values χ^0 as $t \to t^0$, and, consequently, one may speak of a Limit Yield Condition (LYC): $\Phi(\sigma, \chi^0) = 0$.

At the post adaptation stage, the representative stress point in the stress space reaches the yield surface repeatedly, but the stresses do not cause a plastic deformation, and the yield surface does not change. This is possible in perfectly plastic bodies, if the stress path is inside the yield surface, or touches it at some isolated instants. In the bodies with internal variables plastic deformation does not occur under neutral loading in addition.

This Section is devoted to the material models with internal variables. The perfectly plastic bodies are considered in Section 6, the isotropically strain hardening ones are done in Section 7.

So, at the post adaptation stage, the stresses either satisfy the inequality $\Phi(\boldsymbol{\sigma}(t), \boldsymbol{\chi}^0) < 0$, or the equalities:

$$\Phi(\boldsymbol{\sigma}(t),\boldsymbol{\chi}^0) = 0, \quad \text{and} \quad \Phi_{,\sigma}(\boldsymbol{\sigma}(t), \boldsymbol{\chi}^0) : \dot{\boldsymbol{\sigma}} = 0. \tag{7}$$

It is important that equalities (7) are also valid at the time instants (t^*) corresponding to the beginning of unloading, when the representative stress point leaves the yield surface (when the stress point moves along the yield surface), some additional requirements have to be established. Let us consider them.

At the point of departure $\Phi(\boldsymbol{\sigma}(t^*), \boldsymbol{\chi}^0) = 0$, and $\Phi(\boldsymbol{\sigma}(t^* + \Delta t), \boldsymbol{\chi}^0) < 0$, where $\Delta t > 0$ is an arbitrary small increment of t. Accounting for (7), the last inequality can be modified as

$$\dot{\boldsymbol{\sigma}}(t^*+0) : \Phi_{,\sigma\sigma}(\boldsymbol{\sigma}(t^*+0), \boldsymbol{\chi}^0) : \dot{\boldsymbol{\sigma}}(t^*+0) + \Phi_{,\sigma}(\boldsymbol{\sigma}(t^*+0), \boldsymbol{\chi}^0) : \ddot{\boldsymbol{\sigma}}(t^*+0) < 0. \tag{8}$$

Inequality (8) allows to distinguish the instants of departure from those of the neutral loading. However it is equally valid for the instants of departure both at the adaptation and post adaptation stages, that is, irrespective of whether those instants are preceded by plastic deformation, or not. To distinguish between these two opportunities let us remember that $\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}^E(t) + \boldsymbol{\sigma}^r$ where $\boldsymbol{\sigma}^E(t)$ represents the pure elastic response of the reference body to applied loads, and $\boldsymbol{\sigma}^r$ is the residual stress tensor. During neutral loading, for ($t < t^*$) no plastic deformation occurs, therefore $\boldsymbol{\sigma}^r$, $\boldsymbol{\chi}$ = const, and $\dot{\boldsymbol{\sigma}}(t^*-0) = \dot{\boldsymbol{\sigma}}^E(t^*-0)$, $\ddot{\boldsymbol{\sigma}}(t^*-0) = \ddot{\boldsymbol{\sigma}}^E(t^*-0)$.

After departure, for $t > t^*$, also no plastic deformation occurs, so $\dot{\boldsymbol{\sigma}}(t^*+0) = \dot{\boldsymbol{\sigma}}^E(t^*+0)$, $\ddot{\boldsymbol{\sigma}}(t^*+0) = \ddot{\boldsymbol{\sigma}}^E(t^*+0)$. Subsequently at the instants of departure during the post adaptation (stationary) stage

$$\dot{\boldsymbol{\sigma}}(t^*) = \ddot{\boldsymbol{\sigma}}^E(t^*), \quad \text{and} \quad \ddot{\boldsymbol{\sigma}}(t^*) = \ddot{\boldsymbol{\sigma}}^E(t^*). \tag{9}$$

On the other hand, at the adaptation (transient) stage the plastic deformation precedes the instants of departure, $\boldsymbol{\sigma}^r$, $\boldsymbol{\chi}$ are not constant, and equalities (9) are not valid.

Hence, the equalities (9) are specific for the instants of departure at the post adaptation stage.

Now equalities (7), (8) can be modified as follows

$$\Phi(\boldsymbol{\sigma}^E(t^*) + \boldsymbol{\sigma}^r, \boldsymbol{\chi}^0) = 0, \text{and}\, \Phi_{,\sigma}(\boldsymbol{\sigma}^E(t^*) + \boldsymbol{\sigma}^r, \boldsymbol{\chi}^0) : \dot{\boldsymbol{\sigma}}^E(t^*) = 0, \tag{10}$$

$$\dot{\boldsymbol{\sigma}}^E(t^*) : \Phi_{,\sigma\sigma}(\boldsymbol{\sigma}^E(t^*) + \boldsymbol{\sigma}^r, \boldsymbol{\chi}^0) : \dot{\boldsymbol{\sigma}}^E(t^*) + \Phi_{,\sigma}(\boldsymbol{\sigma}^E(t^*) + \boldsymbol{\sigma}^r, \boldsymbol{\chi}^0) : \ddot{\boldsymbol{\sigma}}^E(t^*) < 0. \tag{11}$$

Requirements (10), (11) allow to distinguish between the points of departure at the post adaptation stage and those at the adaptation stage.

If the stress path has a knee at the departure point, requirement (11) has to be replaced by the inequality

$$\Phi_{,\sigma}(\boldsymbol{\sigma}(t^*) + \boldsymbol{\sigma}^r, \boldsymbol{\chi}^0) : \dot{\boldsymbol{\sigma}}^E(t^*+0) < 0. \tag{11$_1$}$$

The existence of the departure points satisfying the requirements (10), (11), or (11_1) is a necessary condition for elastic adaptation. It can be formulated as follows: If elastic shakedown possible for a tentative value of the vector of internal parameters $\boldsymbol{\chi}^*$, then there exist such time instants t^* for which requirements (10), (11), or (11_1) are valid.

4. CONDITION OF INADAPTATION

Conditions (10), (11)/(11$_1$) state that the points of departure at the post adaptation stage are the points of local maximum of the function $\zeta = \Phi(\boldsymbol{\sigma}^E(t)+\boldsymbol{\sigma}^r, \boldsymbol{\chi}^*)$ under fixed values of $\boldsymbol{\sigma}^r$ and $\boldsymbol{\chi}^*$.

It is supposed as usual that $\zeta = \Phi(\boldsymbol{\sigma}, \boldsymbol{\chi}^*)$ is a growing function of $\boldsymbol{\sigma}$. However, due to cyclic nature of loading the function $\zeta = \Phi(\boldsymbol{\sigma}(t), \boldsymbol{\chi}^*)$ has the points of local extremum with respect to time which are determined by the stress path, and, in turn, by the loading program.

The function $\zeta = \Phi(\boldsymbol{\sigma}^E(t)+\boldsymbol{\sigma}^r, \boldsymbol{\chi}^*)$ can have several points of local maximum for fixed values of $\boldsymbol{\sigma}^r$ and $\boldsymbol{\chi}^*$: $\zeta_{\max} = \Phi(\boldsymbol{\sigma}^E(t^*)+\boldsymbol{\sigma}^r, \boldsymbol{\chi}^*)$ where t^* is the related value of t. Depending on $\boldsymbol{\sigma}^r$ the function $\zeta = \Phi(\boldsymbol{\sigma}^E(t)+\boldsymbol{\sigma}^r, \boldsymbol{\chi}^*)$ can reach its absolute (non local) maximum value (denoted maxζ) under different values of t^*.

Choose $\boldsymbol{\sigma}^r$ in such a way that the quantity maxζ would be minimum. The related values of $\boldsymbol{\sigma}^r$ and t^* (denoted $\tilde{\boldsymbol{\delta}}^r$, $\tilde{t}$) are determined by the solution of the problem of mathematical programming:

$$Z = \min_{\boldsymbol{\sigma}^r} \max_{t} \Phi(\boldsymbol{\sigma}^E(t)+\boldsymbol{\sigma}^r, \boldsymbol{\chi}^*) \tag{12}$$

under requirements $\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^r = 0$, and the homogeneous boundary conditions, where $\boldsymbol{\nabla}$ is the vector with components $\partial/\partial x_i$, Z is the corresponding value of ζ, $\boldsymbol{\chi}^*$ is a fixed parameter.

Consider the function $\phi(\boldsymbol{\sigma}^E(t)+\boldsymbol{\sigma}^r, \boldsymbol{\chi}^*)$ and related yield condition $\phi(\boldsymbol{\sigma}^E(t)+\boldsymbol{\sigma}^r, \boldsymbol{\chi}^*) = 0$. According to agreement, if $\phi < 0$ for some values of t and $\boldsymbol{\sigma}^r$, then the corresponding representative stress point $\boldsymbol{\sigma} = \boldsymbol{\sigma}^E(t)+\boldsymbol{\sigma}^r$ is the interior of the yield surface $F = 0$. Hence, if $Z \leqslant 0$ for an admissible value of $\boldsymbol{\chi}^*$, the adaptation is possible because in this case the stress path $\boldsymbol{\sigma} = \boldsymbol{\sigma}^E(t)+\boldsymbol{\sigma}^r$ is either in the interior of the yield surface $\Phi(\boldsymbol{\sigma}, \boldsymbol{\chi}^*) = 0$, or it lies partly on it, or touches it. On the contrary, if $Z > 0$ for a value of $\boldsymbol{\chi}^*$, adaptation is impossible.

Now a sufficient condition for inadaptation can be formulated: Adaptation is impossible for a fixed value of the internal parameters' vector $\boldsymbol{\chi}^*$, if the solution of the problem (12) is positive: $Z > 0$, or if it is not exists.

The above derivation is not restricted by assumptions concerning the yield function, or the internal variables (limited or not limited, for example).

It will be shown below, that the requirement $Z < 0$ is sufficient for adaptation of isotropically strain hardening bodies, and perfectly plastic ones as well.

5. ISOTROPICALLY STRAIN HARDENING BODIES

In the case of isotropically strain hardening bodies the yield function can be represented in the form: $\Phi(\boldsymbol{\sigma}, \chi) = F(\boldsymbol{\sigma})-k(\chi)$ where $k(\chi)$ is the yield stress, and χ is a scalar internal variable named the hardening parameter.

Requirements (10), (11), (11$_1$) do not depend on χ in the case, and take the forms:

$$F(\boldsymbol{\sigma}^E(t)+\boldsymbol{\sigma}^r) = k(\chi), F_{,\boldsymbol{\sigma}}(\boldsymbol{\sigma}^E(t)+\boldsymbol{\sigma}^r):\dot{\boldsymbol{\sigma}}^E = 0, \tag{13}$$

$$\dot{\boldsymbol{\sigma}}^E(t^*):F_{,\boldsymbol{\sigma}\boldsymbol{\sigma}}(\boldsymbol{\sigma}^E(t^*)+\boldsymbol{\sigma}^r):\dot{\boldsymbol{\sigma}}^E(t^*)+F_{,\boldsymbol{\sigma}}(\boldsymbol{\sigma}^E(t^*)+\boldsymbol{\sigma}^r):\ddot{\boldsymbol{\sigma}}^E(t^*) < 0, \tag{14}$$

or

$$F_{,\boldsymbol{\sigma}}(\boldsymbol{\sigma}(t^*)):\dot{\boldsymbol{\sigma}}(t^*+0) < 0. \tag{15}$$

Problem (12) is reduced to the following one

$$Z = \min\max F(\boldsymbol{\sigma}^E(t)+\boldsymbol{\sigma}^r). \tag{16}$$

The value of the hardening parameter corresponding to Z is determined from the equation $k(\chi) = Z$. Since $k(\chi)$ is assumed to be a non decreasing function of χ, the above equation provides us with the minimum value of the hardening parameter χ^* which can be acquired during the adaptation stage. The actual value of the hardening parameter (χ) depends on deformation path. It is not less than χ^*.

The solution of problem (16), if it exists, provides us with the field of residual stresses $\boldsymbol{\sigma}^{r*}$, and with the value of the strain hardening parameter χ^* which satisfy the condition of the Melan theorem extended to isotropically strain hardening bodies (Druyanov and Roman, 1995, 1996): the stress field $\boldsymbol{\sigma}^E(t)+\boldsymbol{\sigma}^{r*}$ satisfies the inequality $F(\boldsymbol{\sigma}^E(t)+\boldsymbol{\sigma}^{r*}) < k(\chi^*+\delta)$ from the time t^* on where δ is an arbitrary small positive number.

Hence, the existence of the solution of problem (16) is a sufficient condition for shakedown of isotropically strain hardening bodies, if, of course, χ^* is not more than the value of the hardening parameter corresponding to the point of material instability.

If the hardening is limited there exists the maximum value of the hardening parameter (χ_m) which is greater than any admissible value of the hardening parameter (χ), or equals it: $\chi \leqslant \chi_m$. If the equation $k(\chi) = F(\boldsymbol{\sigma}^*)$ has no roots, or its roots do not satisfy this inequality, shakedown is impossible.

Thus, the developed method allows not only to obtain the lower estimation for the hardening parameter but also to answer the question of whether the body with limited hardening can shake down to the given loading program, or not.

In the case where bounds of external loads are prescribed, the above condition is sufficient for shakedown, if it is valid for any loading program from the prescribed bounds. Its validity for a loading program lying in the prescribed bounds is necessary.

6. PERFECTLY PLASTIC BODIES

Take the yield condition in the form: $F(\boldsymbol{\sigma}) = k$. As distinct from the strain hardening bodies, in the case of the perfectly plastic ones the representative stress point may only touch the yield surface at some isolated moments during the post adaptation stage. Subsequently, relations (13), (14)/(15) hold again. The difference is that in the case under consideration relations (13) are valid only at some isolated time instants: $t = t^*$.

The existence of the solution of problem (16) is a necessary condition for shakedown. If, overmore, $Z < k$ then the Melan sufficient condition for adaptation is fulfilled, and the body will shakedown to the given loading program.

7. EXAMPLE

Let us consider the structure (Fig. 1) from the previous paper by the authors (Druyanov and Roman, 1996), and determine conditions of adaptation by the method developed in the previous sections.

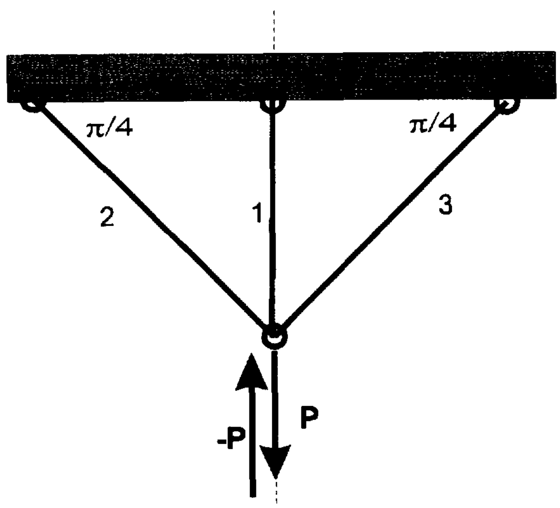

Fig. 1.

The structure consists of three rods of the same cross sections' areas, and of the same material. It is loaded by a variable force P ranging from $-P_1$ to P_2, $P_1 < P_2$. The rods can experience only uniaxial tensile/compressive deformation.

Due to symmetry the strains and stresses in rods 2 and 3 are identical: $\varepsilon_2 = \varepsilon_3$, $\sigma_2 = \sigma_3$. The strains of rods 1 and 2 are connected by the relation: $\varepsilon_1 = 2\varepsilon_2$. The stresses in the rods satisfy the equilibrium equation: $\sigma_1 + \sigma_2\sqrt{2} = p$, where $-p_1 \leqslant p \leqslant p_2$, $p = P/S$, $p_1 = P_1/S$, $p_2 = P_2/S$.

In elastic state $\sigma_1^e = 2\,\sigma_2^e = p(2-\sqrt{2})$.

Let the rods be of a material with bounded isotropically strain hardening. Its yield condition is taken in the form: $F = |\sigma| = k(\chi)$, where $k = k_0 + c\chi^\alpha$, if $\chi \leqslant \chi_m$, and $k = k_m = k_0 + c\chi_m^\alpha = \text{const}$, if $\chi \geqslant \chi_m$. (α, c, k_0 are material parameters, and χ is a hardening parameter defined by the equation: $\dot{\chi} = |\dot{\varepsilon}^p|$, ε^p is the plastic part of rod 1 strain).

Assume that rods 2, 3 remain elastic. Beyond the elastic limit the stress in rod 1 can be represented as $\sigma_1 = \sigma_1^E + \sigma_1^r = p(2-\sqrt{2}) + \sigma_1^r$ where σ_1^r is the residual stress in rod 1. The yield function is: $F(\sigma) = |\sigma| = |\,p(2-\sqrt{2}) + \sigma_1^r\,|$. For a fixed value of σ_1^r the function $F(\sigma)$ reaches its maximum value for $p = p_2$, if $p(2-\sqrt{2}) + \sigma_1^r \geqslant 0$: $\max F = F_2 = p_2(2-\sqrt{2}) + \sigma_1^r$, however, if $p(2-\sqrt{2}) + \sigma_1^r < 0$, the $F(\sigma)$ is maximum for $p = p_1$: $\max F = F_1 = p_1(2-\sqrt{2}) - \sigma_1^r$. Thus the maximum value of F depends on σ_1^r.

$\zeta = \max F$ is minimum if $F_1 = F_2$ that is under $\sigma_1^r = -(p_2 - p_1)(1-\sqrt{2}/2)$. For such the requirement $Z = \min \max F = (p_1 + p_2)(1-\sqrt{2}/2)$. This is the minimal value of the yield stress under which adaptation is possible. The related value of the hardening parameter equals

$$\chi_{\min} = \left(\frac{(p_1 + p_2)(1-\sqrt{2}/2) - k_0}{c}\right)^{1/\alpha}. \tag{19}$$

This value of the strain hardening parameter coincides with that of the limit stress obtained in the previous paper of the authors (Druyanov and Roman, 1996) by another method.

If $\chi_{min} > \chi_m$, shakedown is impossible.

Acknowledgement—The research has been implemented in framework of the Grant from the Israel Science Foundation.

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